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Periodic solutions of the intermediate long-wave equation: a nonlinear superposition principle

A Parker

Department of Mathematics, University of Newcastle upon Tyne, NE1 7RU, UK

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Abstract. Periodic stationary-wave solutions of the intermediate long-wave (ILW) equation are derived using the bilinear transformation method, and a new expression for the dispersion relation is obtained. The class of physically important real-valued solutions is identified. These solutions may be represented as an infinite superposition of solitary-wave *profiles*, a property shared by the related Korteweg-de Vries (KdV) and Benjamin-Ono (BO) equations. This *nonlinear superposition principle*, which has been the subject of various interpretations in the literature, is discussed. The ILW periodic solution approximates to a sinusoidal wave and a solitary wave in the limits of small and large amplitudes, respectively. For intermediate amplitudes the solution can be well approximated by either a sine wave or solitary wave. In the shallow-water (KdV) limit the ILW periodic solution leads to the familiar cnoidal wave, whereas the deep-water (BO) limit yields Benjamin's periodic wave. A previously unknown expression for the cnoidal-wave dispersion relation in terms of theta functions is obtained. The controversy surrounding the periodic solutions of the ILW equation reported by other authors is examined in the light of the results reported here. The 'correct' solution (which turns out to be *complex*-valued) is derived as a limit of the more general stationary periodic solution.

1. Introduction

The intermediate long-wave (ILW) equation is a weakly nonlinear integro-differential equation which describes the propagation of long internal gravity waves in a stratified fluid of finite depth [1-3]. The equation has also been used to model large-scale wave motions in both the atmosphere and oceans [4-6], nonlinear waves in shear flows [7] and, more recently, the *dead water phenomenon* [8] first reported by Ekman [9].

The ILW equation is a special case of the Whitham equation [10]

$$u_t(x, t)+Cu(x, t)u_x(x, t)+\frac{\partial}{\partial x}\int_{-\infty}^{\infty} u(x', t)G(x'-x)\,\mathrm{d}x'=0 \tag{1a}$$

$$G(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \tag{1b}$$

which describes weakly nonlinear waves in a dispersive medium. The coefficient C is a constant which characterizes the nonlinearity of the system. The integral in (1a) (which is to be interpreted as a Cauchy principal value as necessary) represents the dispersive effects, where $c(k)$ denotes the 'internal' phase speed and $c(0)=c_0$ (a positive constant). For a two-layer fluid of total depth D, with a thin thermocline located at a depth d ($D \gg d$), the appropriate expression for $c(k)$ is given by Phillips [11] as

$$c(k)=c_0\left[1-\tfrac{1}{2}kd\left(\coth(kD)-\frac{1}{kD}\right)\right]. \tag{2}$$

Substituting equation (2) into (1) we obtain the dimensional form of the ILW equation [2, 3]

$$u_t + c_0 u_x + Cuu_x + \frac{c_0 d}{2D}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{u(x', t)\,\mathrm{sgn}(x'-x)}{\exp(\pi|x'-x|/D)-1}\,\mathrm{d}x' = 0. \tag{3}$$

In the shallow-water limit, $D \to 0$, equation (2) gives $c(k) \sim c_0(1 - k^2 dD/6)$ and the ILW (3) reduces to the well known Korteweg–de Vries (KdV) equation [12] (see section 9), whereas the deep-water limit, $D \to \infty$, yields $c(k) \sim c_0(1 - |k|d/2)$ and the ILW (3) reduces to the Benjamin–Ono (BO) equation [13–15] (see section 10).

The ILW equation, like the related BO and KdV equations, exhibits a balance between the effects of nonlinearity and dispersion which leads to the existence of waves of permanent form. Stationary solutions of the KdV equation, in the form of sech^2 solitary waves and periodic cnoidal waves, are well known and were first derived by Korteweg and de Vries in 1895 [16]. The algebraic (Lorentzian) solitary wave and periodic stationary-wave solution of the BO equation were first obtained by Benjamin [13]. The solitary-wave and multisoliton solutions of the ILW were first reported by Joseph [1, 2], and subsequently obtained by Chen and Lee [17] and Matsuno [18] (using Hirota's bilinear transformation method) and by Kodama *et al* [19] (via the inverse scattering transform).

The derivation of the ILW equation from the basic fluid dynamical equations assumes that $u(x, t)$ has a classical Fourier transform and that u vanishes a $|x| \to \infty$. Nevertheless, we may reasonably enquire as to whether equation (3) admits a spatially periodic solution. The numerical work of Kubota *et al* [3] indicates that such solutions exist; their resulting waveforms and phase speeds were found to agree with the known periodic solutions of the KdV and BO equations (with the same period) in the respective limits. Recently, Miloh [20] has derived an analytic expression for the periodic stationary-wave solution of the ILW equation (3), which he claims (mistakenly, as it turns out) to report for the first time. His method of solution is to assume that a periodic solution can be represented as a doubly 'infinite sum of spatially repeated solitons' and then to show, by direct substitution into (3), that it is a solution for a suitably chosen dispersion relation and arbitrary integration constant. This remarkable result, that a nonlinear periodic solution can be expressed as an infinite superposition of identical 'solitons', has been demonstrated for a number of nonlinear evolution equations including the KdV equation [21–23] and, more recently, for the BO equation [24]. However, the interpretation given to the infinite series by these authors, namely that each series is a sum of repeated solitons (more correctly termed solitary waves) of the respective evolution equation, is technically incorrect and an alternative viewpoint is presented at the end of this paper.

The periodic solution obtained by Miloh [20] had, in fact, already been reported by Zaitsev [25] (albeit for a dimensionless form of the ILW equation). Interestingly, in the same article, Zaitsev shows that periodic stationary waves in one spatial dimension can be deduced for a wide class of nonlinear evolution equations by superposing an infinite number of 'solitary waves'.

Other attempts to find periodic solutions of the ILW equation have been made, most notably by Joseph and Egri [2], Chen and Lee [17] and Nakamura and Matsuno [26]. However, Miloh [20] and Ablowitz *et al* [27] claim that the analytic solutions obtained by these authors are incorrect; the 'correct' solution, as given by the latter authors, is complex and appears, therefore, to have no physical significance. Certainly, the solution procedure adopted by Joseph and Egri and Chen and Lee can be criticized for being somewhat heuristic (inasmuch as it consists of formally replacing the real wavenumber in the ILW solitary-wave solution by a pure imaginary one) and does indeed lead to an incorrect solution (see section 11). Miloh [20] appears to ascribe (mistakenly in our view) this same method of solution to Nakamura and Matsuno [26] whereas they, in fact, take an entirely different approach based on theta functions. Indeed, Miloh's criticism is ill-founded since in their discussion Nakamura and Matsuno [26] observe that the periodic solution of Joseph (and others) has essentially one free parameter, whereas their own solution contains two independent parameters. They point out that, as a consequence, the former solutions lead to divergent, non-physical solutions in the shallow-water KdV limit (a fact also noted by Chen and Lee), unlike their own solution which relates smoothly to both the KdV and BO periodic waves.

In the present paper we will demonstrate that the bilinear transformation method used by Nakamura and Matsuno does indeed lead to correct periodic stationary-wave solutions of the ILW (3). Further, we shall derive the 'corrected' version of the periodic solution proposed by Joseph and Egri [2] and Chen and Lee [17]. This solution turns out to be complex-valued and is given by Ablowitz *et al* [27], but without any indication as to how it can be derived. The advantage of our approach resides in the fact that our starting point is a complex-valued periodic solution of the ILW from which we are able to derive the latter solution as a particular limiting case (see section 11). Nakamura and Matsuno [26] chose to use a somewhat general definition of a theta function and their article is confined to showing that it solves the bilinear form of the ILW equation. In particular, their resulting expression for the important dispersion relation is complicated and unwieldy, involving, as it does, a ratio of two infinite series of hyperbolic functions. For our own part, we will use a specific theta function: this leads to a new and remarkably simple expression for the dispersion relation which readily lends itself to perturbation theory and numerical calculation. Not least, by proceeding to the KdV limit, we will also obtain an expression for the cnoidal-wave dispersion relation which does not appear to have been reported elsewhere.

Ablowitz *et al* [27] give the conditions under which the solution to the bilinear form of the ILW equation yields a solution of the ILW equation proper; these appear to have been overlooked by Nakamura and Matsuno [26]. As it turns out, these conditions are precisely those that are required to ensure that our own solutions are well defined and analytic. Curiously, for the important class of real periodic solutions, Miloh [20] and Zaitsev [25] both state the correct condition in the absence of any discussion of the convergence of their series solutions.

The periodic solution of the ILW equation has a natural parametrization in terms of the nome of the theta function. A series expansion in the nome shows that the solution can be represented as an infinite superposition of 'solitons' which recovers the expressions obtained by both Miloh [20] and Zaitsev [25]; this is the appropriate perturbation series for exploring the periodic solution when nonlinear effects are at their strongest. On the other hand, its Fourier series representation yields a perturbation series in the complementary nome which allows us to examine the periodic solution in the small-amplitude linear-wave regime. By using these series we can deduce the known periodic solutions of the KdV equation (cnoidal wave) and BO equation (Benjamin's solution) in the shallow-water and deep-water limits, respectively.

2. Mathematical preliminaries

We introduce the transformation

$$x \to \left(\frac{c_0 d}{2}\right)x \qquad t \to \left(\frac{c_0 d}{2}\right)t \qquad u \to \frac{2u - c_0}{C} \tag{4}$$

and define the parameter $\lambda > 0$ by

$$\lambda = c_0 d/2D \tag{5}$$

which characterizes the relative depths of the two fluid layers.

Then, under the scaling (4), the ILW (3) is reduced to the dimensionless form

$$u_t + 2uu_x + G[u]_{xx} = 0 \tag{6}$$

$$G[u] = \frac{\lambda}{2} \int_{-\infty}^{\infty} u(x', t)\{\coth \tfrac{1}{2}\pi\lambda(x'-x) - \operatorname{sgn}(x'-x)\}\, \mathrm{d}x' \tag{7a}$$

$$= \lambda \int_{-\infty}^{\infty} \frac{u(x', t)\operatorname{sgn}(x'-x)}{\exp(\pi\lambda|x'-x|) - 1}\, \mathrm{d}x'. \tag{7b}$$

The form of the equation given by (6) and (7a) is that used by Nakamura and Matsuno [26]. We note that G is a linear operator, and the equivalent expression (7b) shows that $G[u_0] = 0$ if $u_0 = \text{constant}$.

We shall also find it convenient to write equation (6) in the slightly different form

$$u_t + 2uu_x + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} u(x', t)H(x'-x)\, \mathrm{d}x' = 0 \tag{8}$$

$$H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{c}(k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k \qquad \bar{c}(k) = \lambda\left[1 - \frac{k}{\lambda}\coth\left(\frac{k}{\lambda}\right)\right]. \tag{9}$$

This is essentially the form of the equation discussed by Zaitsev [25] and will prove useful for analysing the KdV and BO limits.

The ILW equation (6) is invariant under the Galilean transformation

$$x' = x - 2u_0 t \qquad t' = t \qquad U = u - u_0 \qquad u_0 = \text{constant}. \tag{10}$$

Thus, if $U(x, t)$ is a solution of the ILW equation (6), then so too is $u(x, t) = u_0 + U(x - 2u_0 t, t)$. In particular, if $U(\xi)$, $\xi = x - vt$, is a stationary solution of the ILW equation with phase speed $v = \text{constant}$, then $u = u_0 + U(\xi - 2u_0 t)$ is also a stationary solution with augmented speed $c = 2u_0 + v$.

3. Periodic solutions

Under the nonlinear dependent-variable transformation

$$u(x, t) = u_0 + \mathrm{i}\partial_x \ln[f_+(x, t)/f_-(x, t)] \qquad u_0 = \text{constant} \tag{11}$$

where (following the notation of Ablowitz *et al* [27] and Matsuno [28]) $f_\pm(x, t) = f(x \pm \mathrm{i}/\lambda, t)$ and ∂_x denotes the partial derivative with respect to x, equation (6) can be written in its bilinear form [17, 18]

$$[\mathrm{i}D_t + \mathrm{i}(\lambda + 2u_0)D_x - D_x^2 + B]f_+ \cdot f_- = 0. \tag{12}$$

Here, B is an arbitrary integration 'constant' (possibly dependent on time) and D_t, D_x are the usual bilinear operators defined by (see e.g. [29])

$$D_t^m D_x^n a(x, t) \cdot b(x, t) \equiv (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n a(x, t) b(x', t')|_{x=x', t=t'}.$$

If we use the properties of the bilinear operator, $D_x^m a\cdot b=(-1)^m\, b\cdot a$ and $\exp(\varepsilon D_x)a(x)\cdot b(x)=a(x+\varepsilon)b(x-\varepsilon)$, then it can be shown [28] that the bilinear equation (12) can be re-expressed as

$$F(D_t, D_x)f\cdot f=0 \tag{13a}$$

$$F(D_t, D_x)\equiv \mathrm{i}\{D_t+(\lambda+2u_0)D_x\}\sinh(\mathrm{i}\lambda^{-1}D_x)+(D_x^2-B)\cosh(\mathrm{i}\lambda^{-1}D_x). \tag{13b}$$

We seek a solution of (13) in the form of the theta function θ_3,

$$f(x,t)=\theta_3(z,q)=\sum_{n=-\infty}^{\infty} q^{n^2}\,\mathrm{e}^{2\mathrm{i}nz} \qquad q=\mathrm{e}^{\mathrm{i}\pi\tau} \tag{14}$$

where the phase variable is given by $z=px+\omega t+\alpha$, with p, ω, α arbitrary (possibly complex) parameters at this stage. We remark that $\theta_3(z,q)$ is periodic in z with period π and is an entire function provided the nome q is such that $0<|q|<1$ (i.e. $\mathrm{Im}(\tau)>0$).

Substituting (14) into (13), we obtain the residual equation

$$\tilde{F}_0\theta_3(2z,q^2)+\tilde{F}_1 q^{-1/2}\theta_2(2z,q^2)=0$$

where

$$\tilde{F}_m=\sum_{n=-\infty}^{\infty} F[2\mathrm{i}(2n-m)\omega, 2\mathrm{i}(2n-m)p]q^{n^2+(n-m)^2} \quad m=0,1 \tag{15}$$

and

$$\theta_2(z,q)=\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}\,\mathrm{e}^{\mathrm{i}(2n+1)z}.$$

It follows that $f=\theta_3$ is an exact solution of (13) provided that $\tilde{F}_0=\tilde{F}_1=0$, whence, using (15) and the functional form of F given by (13b), we obtain the pair of simultaneous linear equations in ω and B

$$\lambda[\omega+(\lambda+2u_0)p]A_0'-\lambda^2p^2A_0''-A_0B=0 \tag{16}$$

$$\lambda[\omega+(\lambda+2u_0)p]A_1'-\lambda^2p^2A_1''-A_1B=0 \tag{17}$$

where the quantities A_0, A_1 are defined by

$$A_0(p;q,\lambda)=\sum_{n=-\infty}^{\infty} q^{2n^2}\cosh(4np/\lambda)=\theta_3(2\mathrm{i}\gamma,q^2) \qquad \gamma=p/\lambda \tag{18}$$

$$A_1(p;q,\lambda)=\sum_{n=-\infty}^{\infty} q^{n^2+(n-1)^2}\cosh[2(2n-1)p/\lambda]=q^{1/2}\theta_2(2\mathrm{i}\gamma,q^2). \tag{19}$$

Here, the prime denotes differentiation with respect to the wavenumber p.

Solving equations (16) and (17) for ω and B we find that

$$\omega(p;q,\lambda)=-(\lambda+2u_0)p+\lambda p^2\frac{A_0A_1''-A_0''A_1}{A_0A_1'-A_0'A_1} \tag{20}$$

$$B(p;q,\lambda)=\lambda^2p^2\frac{A_0'A_1''-A_0''A_1'}{A_0A_1'-A_0'A_1}. \tag{21}$$

We note that the dispersion relation (20) can be rewritten in the more compact form

$$\omega=-(\lambda+2u_0)p+\lambda p^2\partial_p\ln W(A_0,A_1) \tag{22}$$

where $W(A_0,A_1)=A_0A_1'-A_0'A_1$ is the Wronskian of A_0 and A_1.

Although the constant of integration B has no dynamical significance, it will be demonstrated later that $B \neq 0$ which ensures the compatibility of equations (16) and (17). Indeed, the usual 'soliton' boundary conditions, $u \to 0$ as $|x| \to \infty$, imply that $B = 0$ in the bilinear equation (13); in this case, the bilinear operator (13b) satisfies the Hirota conditions [30] thereby guaranteeing the existence of solitary-wave and soliton solutions. We conclude that the non-vanishing of B plays a crucial role in the periodic problem, though, as we shall see (section 6), $B \to 0$ as $q \to 0$ which is the solitary-wave limit for the periodic solution.

Thus, $f(x, t) = \theta_3(z)$, together with the transformation (11) and the dispersion relation (22), would appear to give an exact periodic solution of the ILW (6); Nakamura and Matsuno [26] call this the *one-periodic wave solution*† which is, in general, complex-valued. However, it is not immediately apparent that a solution to the bilinear equation (12) (or, equivalently, equation (13)) yields a valid solution of the ILW (6). To do so, it turns out that $f(x, t)$ must satisfy certain analyticity conditions (Ablowitz *et al* [27]) which essentially ensure that the bilinear form of the ILW (6) has the expression given in (12). Nakamura and Matsuno [26] omitted to address these validity requirements in their paper. We shall consider the question of whether $f(x, t) = \theta_3(z)$ satisfies these conditions (which we shall, henceforth, refer to as the 'Ablowitz conditions') in section 5, where we will also identify the important class of real periodic solutions.

4. The dispersion relation

Substituting the series (18) and (19) into (20), and collecting like powers of q, yields a complicated and unwieldy series expansion for ω which is of little practical use (see e.g. Nakamura and Matsuno [26]). However, by noting that $D_x a(x) \cdot b(x) = -W(a, b)$, (22) can be reformulated as

$$\omega = -(\lambda + 2u_0)p + \lambda p^2 \partial_p \ln\{D_p A_0 \cdot A_1\}. \tag{23}$$

Then, using the bilinear property,

$$D_x^n \cosh(p_1 x) \cdot \cosh(p_2 x) = \tfrac{1}{2}[(p_1 - p_2)^n \sinh(p_1 + p_2)x + (p_1 + p_2)^n \sinh(p_1 - p_2)x]$$

for n an odd integer, we find that

$$D_p A_0 \cdot A_1 = \frac{2}{\lambda} \sum_{n,m=-\infty}^{\infty} (2n - 2m + 1) q^{2n^2 + m^2 + (m-1)^2} \sinh[2(2n + 2m - 1)\gamma]. \tag{24}$$

The summations in (24) can be uncoupled by setting $r = n + m$, $s = n - m$, so that

$$D_p A_0 \cdot A_1 = \frac{2q^{1/2}}{\lambda} \sum_{(r,s)} (2r - 1) q^{(r-1/2)^2 + (s+1/2)^2} \sinh[2(2r - 1)\gamma] \tag{25}$$

the summation ranging over all pairs of even integers and all pairs of odd integers (positive, negative and zero). If we now let $(r, s) = (2k, 2l)$ and $(2k+1, 2l+1)$ in (25), then, after a little manipulation, we find that

$$D_p A_0 \cdot A_1 = 2A(q, \lambda) \sum_{k=-\infty}^{\infty} q^{(2k-1/2)^2} \sinh[2(4k - 1)\gamma] \tag{26}$$

where the coefficient A is independent of p.

† These authors use a general expression for the theta function

$$\theta(\eta|\tau) = \sum_{n=-\infty}^{\infty} \exp(n\eta + n^2\tau) \qquad \eta = \mathrm{i}(kx + \omega t + \eta_0) \qquad \tau < 0.$$

The series (26) can now be summed using the theta function

$$\theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)^z$$

to yield

$$D_p A_0 \cdot A_1 = \mathrm{i}A(q, \lambda)\theta_1(2\mathrm{i}\gamma, q). \tag{27}$$

Substituting (27) into (23), we finally deduce the remarkably compact expression for the dispersion relation

$$\omega = -(\lambda + 2u_0)p + 2\mathrm{i}p^2 \frac{\theta_1'(2\mathrm{i}\gamma, q)}{\theta_1(2\mathrm{i}\gamma, q)} \qquad \gamma = p/\lambda \tag{28}$$

where the prime denotes the derivative of $\theta_1(z, q)$ with respect to z. We note that the explicit form of $A(q, \lambda)$ is not important, since it vanishes on taking the logarithmic derivative in (23). However, it is not difficult to show that A can also be written in terms of θ_1 as $A = q^{1/2}\theta_1'(0, q)/\lambda$. It is rewarding to note that the dispersion relation (28) bears more than a passing resemblance to that for the periodic (cnoidal-wave) solution of the related KdV equation [30]. The importance of the dispersion relation in the form (28) rests in the fact that it readily lends itself to perturbation theory and numerical computation, a significant advantage over the previously reported expression [26].

Likewise, we can show that the integration constant B has an expression in terms of theta functions, namely

$$B(\gamma; q) = -p^2 \left[\frac{\theta_1''(2\mathrm{i}\gamma, q)}{\theta_1(2\mathrm{i}\gamma, q)} - \frac{\theta_1'''(0, q)}{\theta_1'(0, q)}\right] \qquad \gamma = p/\lambda. \tag{29}$$

5. Analyticity conditions and real periodic solutions

We shall find it convenient to let $z \to \frac{1}{2}z$ (i.e. $p \to \frac{1}{2}p$, $\omega \to \frac{1}{2}\omega$, etc) in (14), and we therefore consider the periodic solution given by setting $f(x, t) = \theta_3(\frac{1}{2}z, q)$. With the nome q understood in what follows, the periodic solution (11) becomes

$$u(x, t) = u_0 + \mathrm{i}\partial_x \ln\left\{\frac{\theta_3[\frac{1}{2}(z - \mathrm{i}\gamma)]}{\theta_3[\frac{1}{2}(z + \mathrm{i}\gamma)]}\right\} \qquad \gamma = p/\lambda \tag{30}$$

with dispersion relation given by (28) as

$$\omega = -(\lambda + 2u_0)p + \mathrm{i}p^2\theta_1'(\mathrm{i}\gamma)/\theta_1(\mathrm{i}\gamma). \tag{31}$$

The integration constant (29) is replaced by

$$B(\gamma; q) = -\frac{p^2}{4}\left[\frac{\theta_1''(\mathrm{i}\gamma)}{\theta_1(\mathrm{i}\gamma)} - \frac{\theta_1'''(0)}{\theta_1'(0)}\right]. \tag{32}$$

It can be shown that B has the series expansion

$$B(\gamma; q) = -8p^2 \sinh^2\gamma \sum_{n=1}^{\infty} \frac{nq^{2n}(1+q^{2n})}{(1-q^{2n})(1-2q^{2n}\cosh 2\gamma + q^{4n})} \tag{33}$$

which shows that $B \neq 0$, as required (recall that $0 < |q| < 1$).

Let us observe that the function

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln \theta(z) = \frac{\theta'(z)}{\theta(z)} \tag{34}$$

where $\theta(z)$ denotes any of the four theta functions, $\theta_i(z)$ $(i=1,2,3,4)$, is periodic with real period π, and has the complex quasiperiod $\pi\tau$ since

$$\frac{\theta'(z+\pi\tau)}{\theta(z+\pi\tau)}=\frac{\theta'(z)}{\theta(z)}-2\mathrm{i}.$$

It follows that the solution (30) is biperiodic in z with periods 2π and $2\pi\tau$ whose ratio is complex definite $(\mathrm{Im}(\tau)>0)$. Thus the general solution (30) is an elliptic function which is biperiodic in the spatial variable x with periods $2\pi/p$ and $2\pi\tau/p$. These periods will be either real or complex depending upon our choice of the parameters p and τ. Moreover, since $\theta(z)$ has either even or odd parity in z, the function defined in (34) has odd parity. It is now straightforward to show that the solution (30) is invariant under the parity transformation $p\to -p$ (modulo a phaseshift $\alpha\to-\alpha$).

The periodic solution (30) is, in general, complex-valued. Since we shall be interested in physical applications, we must eventually choose the free parameters u_0, p, α and q (or τ) so as to obtain real solutions. With this in mind we shall, henceforth, let u_0 be real and take $\tau=\mathrm{i}s$ $(s>0)$ to be pure imaginary, whereupon $q=\exp(-\pi s)$ is such that $0<q<1$. To ensure that we get a real spatial period, two choices for the wavenumber p are now available.

5.1. p real

Because of parity invariance we may, without loss of generality, take $p>0$. Then $\gamma=p/\lambda>0$ and, using the complex conjugate property

$$\left[\frac{\theta'(z,q)}{\theta(z,q)}\right]^*=\frac{\theta'(z^*,q^*)}{\theta(z^*,q^*)}\qquad \theta=\theta_i\quad(i=1,2,3,4) \tag{35}$$

we find that ω (equation (31)) is real. Thus, the phase variable $z=px+\omega t+\alpha$ is real or complex depending on whether the phaseshift α is real or complex, respectively.

In this case, the solution (30) may be written in the form

$$u(x,t)=u_0+\tfrac{1}{2}\mathrm{i}p\left[\frac{\theta_3'[\frac{1}{2}(z-\mathrm{i}\gamma)]}{\theta_3[\frac{1}{2}(z-\mathrm{i}\gamma)]}-\frac{\theta_3'[\frac{1}{2}(z+\mathrm{i}\gamma)]}{\theta_3[\frac{1}{2}(z+\mathrm{i}\gamma)]}\right]\qquad p,\gamma>0 \tag{36}$$

with the real dispersion relation given by (31). The solution is spatially periodic (in x) with real period $\sigma=2\pi/p$.

The function $\theta_3(z)$ has simple zeros at $z=(m+\frac{1}{2})\pi+(n+\frac{1}{2})\pi\tau$ (m, n integers) and so the solution (36) has a lattice of simple poles which must be avoided. It is straightforward to show that the solution (36) is well defined and analytic in the strip $|\mathrm{Im}(z)|<\pi s-\gamma$, and we therefore obtain the analyticity condition

$$|\mathrm{Im}(\alpha)|<\pi s-\gamma\qquad 0<\gamma<\pi s. \tag{37}$$

It can be shown that (37) is precisely equivalent to the Ablowitz conditions (see section 3) which ensure that $u(x,t)$, equation (36), is a solution of the ILW (6).

Incidentally, the solution (36) can be expressed in terms of Jacobi's zeta function as

$$u(x,t)=u_0+\frac{\mathrm{i}K(k)p}{\pi}\left\{Z\left[\frac{K(k)}{\pi}(z-\mathrm{i}\gamma),k\right]-Z\left[\frac{K(k)}{\pi}(z+\mathrm{i}\gamma),k\right]\right\}$$

where $K(k)>0$ denotes the complete elliptic integral of the first kind with modulus k $(0<k<1)$.

It is evident from equations (35) and (36) that the solution $u(x, t)$ is real-valued only if α is real (i.e. z is real); otherwise it is complex-valued. Thus, for real stationary periodic solutions, which may be used to represent physical waves, the analyticity condition (37) reduces to $0<\gamma<\pi s$.

5.2. *p purely imaginary*

We now let $p \to \mathrm{i}p$ where we may again take $p>0$ because of parity invariance. Then, letting $\gamma \to \mathrm{i}\gamma$, $\gamma = p/\lambda > 0$, and using equation (35), we find that ω (equation (31)) is pure imaginary. We therefore transform $\omega \to \mathrm{i}\omega$, whereupon (31) yields the dispersion relation

$$\omega = -(\lambda + 2u_0)p + p^2 \frac{\theta_1'(\gamma)}{\theta_1(\gamma)}. \tag{38}$$

If we replace α by $\mathrm{i}\alpha$, then the phase variable $z \to \mathrm{i}z$, $z = px + \omega t + \alpha$, and again z is real or complex according to whether α is real or complex, respectively.

The solution (30) now takes the form

$$u(x, t) = u_0 + \mathrm{i}\partial_x \ln\left[\frac{\theta_3[\frac{1}{2}(\mathrm{i}z+\gamma)]}{\theta_3[\frac{1}{2}(\mathrm{i}z-\gamma)]}\right] \qquad p, \gamma > 0 \tag{39}$$

which has a real spatial period (in x) given by $\sigma = 2\pi s/p$.

The simple poles of $u(x, t)$ can be avoided provided that we restrict z to the domain $|\mathrm{Im}(z)| < \pi - \gamma$. Thus, (39) yields a well defined, analytic solution if the analyticity condition

$$|\mathrm{Im}(\alpha)| < \pi - \gamma \qquad 0 < \gamma < \pi \tag{40}$$

is satisfied. It is again easy to show that (40) is equivalent to the Ablowitz conditions which ensure that (39) is a solution of the ILW (6).

Expanding the solution (39) as

$$u(x, t) = u_0 - \tfrac{1}{2}p\left[\frac{\theta_3'[\frac{1}{2}(\mathrm{i}z+\gamma)]}{\theta_3[\frac{1}{2}(\mathrm{i}z+\gamma)]} - \frac{\theta_3'[\frac{1}{2}(\mathrm{i}z-\gamma)]}{\theta_3[\frac{1}{2}(\mathrm{i}z-\gamma)]}\right]$$

and using equation (35), it is evident that $u(x, t)$ is real-valued only if the phaseshift α is real. In this case, the analyticity condition (40) reduces to

$$0 < \gamma < \pi. \tag{41}$$

In summary, we see that it is possible to obtain valid stationary periodic solutions $u(x, t)$ of the ILW (6) in two different ways: by choosing the phase variable z to be either 'real' or 'pure imaginary' (modulo the arbitrary phaseshift α) in (30). Both choices lead to physically important real solutions by choosing the phaseshift α appropriately. It is therefore important to consider whether these two solutions are in any sense distinct. In fact, it turns out that the two solutions are equivalent to one another under a suitable transformation of parameters. To demonstrate their equivalence one uses Jacobi's modular transformation for theta functions; although straightforward, this requires some tedious technical manipulation. In particular, of course, the real solutions in each case represent physical waves that are dynamically equivalent in every respect.

6. A nonlinear superposition principle: solitary-wave limit

Let us consider the periodic solution of the ILW (6) given by equation (39). Since the phaseshift α is arbitrary, we may transform $\alpha \to \alpha + \pi s$ without affecting the dispersion relation (38) or the constant of integration B. If we note that $\mathrm{i}z \to \mathrm{i}z + \pi\tau$ ($\tau = \mathrm{i}s$) and use the quasiperiodicity property $\theta_2(z) = \mu\theta_3(z + \frac{1}{2}\pi\tau)$, $\mu = q^{1/4}\exp(\mathrm{i}z)$, then the solution (39) becomes

$$u(x, t) = u_0 + \mathrm{i}\partial_x \ln\left[\frac{\theta_2[\frac{1}{2}(\mathrm{i}z+\gamma)]}{\theta_2[\frac{1}{2}(\mathrm{i}z-\gamma)]}\right] \tag{42}$$

where the periodicity factor $\mu(z)$ has been eliminated by the logarithmic derivative. Moreover, the analyticity condition (40) is unaffected by the transformation. The solution in the form (42) will allow us to recover the (real) periodic solutions obtained by Miloh [20] and Zaitsev [25].

We now make use of the identity

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\theta_2(z) = \frac{\theta_2'(z)}{\theta_2(z)} = \mathrm{i}\sum_{n=-\infty}^{\infty}\tanh[\mathrm{i}(z - n\pi\tau)] \tag{43}$$

which we readily deduce from the infinite product expansion

$$\theta_2(z, q) = 2q^{1/4}\cos z \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n}\cos 2z + q^{4n}).$$

Substituting (43) into (42), and making use of the elementary hyperbolic identity

$$\tanh A - \tanh B = \frac{2\sinh(A-B)}{\cosh(A-B)+\cosh(A+B)} \tag{44}$$

we find that

$$u(x, t) = u_0 + p\sum_{n=-\infty}^{\infty}\frac{\sin\gamma}{\cosh(z - 2n\pi s) + \cos\gamma} \qquad z = px + \omega t + \alpha \tag{45}$$

with the dispersion relation given by (38).

If we take α to be real, then (45) is a real periodic stationary-wave solution of the ILW (6) (with wavelength $\sigma = 2\pi s/p$) provided that the analyticity condition (41) is satisfied. It is reassuring to note that (45) recovers the (real) periodic solution obtained by Zaitsev [25], under a suitable identification of parameters.

With the aid of the series representation (45), we are now able to give an interpretation to the real periodic solution $u(x, t)$ as an infinite sum of regularly spaced 'solitary waves'. The solitary-wave (one-soliton) solution of the ILW equation is well known and has been derived independently by Joseph [1], Chen and Lee [17], Matsuno [18] and Kodama *et al* [19]. For the ILW (6), it is given by

$$u_s(x, t) = \frac{p\sin\gamma}{\cosh(px + \omega_s t + \alpha) + \cos\gamma} \qquad 0 < \gamma < \pi \tag{46}$$

$$\omega_s = -\lambda p + p^2\cot\gamma \tag{47}$$

which satisfies the usual 'soliton' boundary conditions $u \to 0$ as $|x| \to \infty$. (It is easily seen that the solitary wave satisfying the more general boundary conditions, $u \to u_0$ as $|x| \to \infty$, can be deduced from (46) and (47) using the Galilean transformation (10).)

We remark that, with $0 < \gamma < \pi$, the solitary-wave speed

$$c_s = \lambda[1 - \gamma\cot\gamma]$$

is positive, i.e. the solitary wave is unidirectional. This is in contradistinction to the assertion of Miloh [20] that the solitary wave may travel both to the left and right.

Comparison of (45) and (46) shows that the periodic solution of the ILW equation can be represented as an infinite superposition of equally spaced solitary-wave *profiles* whose crests are centred at $z=0, \pm 2\pi s, \pm 4\pi s, \ldots$. This remarkable property, whereby a nonlinear periodic wave can be expressed as an exact sum of solitary-wave shapes, is well known for the KdV equation. It was first reported in 1975 by Toda [21] who demonstrated that the cnoidal wave can be written as a doubly infinite sum of repeated sech^2 solitary-wave profiles. A large number of nonlinear evolution equations have since been shown to possess this same property [23-25]. Because of its wide applicability, this property may legitimately be regarded as a *nonlinear superposition principle.* However, caution must be exercised when interpreting the principle: we do not have a superposition of solitary-wave solutions in the accepted sense of linear theory. This is because the speed of the periodic wave differs, in general, from that of the solitary wave whose shape is replicated to generate the periodic solution. Thus, we do not have a superposition of solitary waves *per se*, but only a superposing of their shapes. This observation appears to have been overlooked in much of the literature, and the question of how we should interpret this property is considered more fully in section 12.

To be precise in the present instance, we can use the series [31]

$$\frac{\theta_1'(z)}{\theta_1(z)} = \cot z + 4 \sin 2z \sum_{n=1}^{\infty} \frac{q^{2n}}{1-2q^{2n}\cos 2z + q^{4n}} \qquad (z \neq m\pi + n\pi\tau) \tag{48}$$

to write the dispersion relation (38) as

$$\omega = \omega_s - 2u_0 p + 4p^2 \sin 2\gamma \sum_{n=1}^{\infty} \frac{q^{2n}}{1-2q^{2n}\cos 2\gamma + q^{4n}} \qquad 0<\gamma<\pi \tag{49}$$

(where we note that the analyticity condition (41) ensures the convergence of this last series). Recalling that $q=\exp(-\pi s)$, $s>0$, the periodic wave (45) has speed $c=-\omega/p$ which can be expressed as

$$c = c_s + \Delta c \qquad \Delta c = 2\left\{u_0 - p\sin 2\gamma \sum_{n=1}^{\infty} [\cosh(2n\pi s) - \cos 2\gamma]^{-1}\right\}. \tag{50}$$

This shows that, in general, $c \neq c_s$. However, because u_0 can be chosen arbitrarily, the particular choice

$$u_0 = p \sin 2\gamma \sum_{n=1}^{\infty} [\cosh(2n\pi s) - \cos 2\gamma]^{-1} \qquad 0<\gamma<\pi \tag{51}$$

ensures that $\Delta c = 0$. Because the wavespeeds of the periodic wave and solitary wave (46) now coincide, Zaitsev [25] interprets the summation (45) as a true linear superposition of solitary waves in this case. But this conclusion is clearly mistaken since (51) shows that, in general, $u_0 \neq 0$; see solution (45). However, for the value of $\gamma = \frac{1}{2}\pi$ only, we see that $u_0 = 0$: for this very special choice, (45) now becomes

$$u(x,t) = p \sum_{n=-\infty}^{\infty} \operatorname{sech}(z - 2n\pi s) \qquad z = px + \omega t + \alpha \tag{52}$$

which is a linear superposition of solitary waves in the accepted sense, each one being of the form

$$u_s(x, t) = p \operatorname{sech} z \qquad \omega_s = -\lambda p = -2p^2/\pi. \tag{53}$$

Incidentally, (53) shows that the ILW equation possesses a solitary wave which has a shape identical to that of the 'sech'-type solitary wave of both the modified KdV equation and the envelope solitary wave of the nonlinear Schrödinger equation.

Figures 1 and 2 illustrate the superposition principle for the values $\gamma = \frac{1}{4}\pi$ and $\gamma = 3\pi/4$, respectively; we have put $u_0 = 0$, $p = 1$ in both cases. Each figure shows the resulting periodic wave for the three parameter values $s = 0.5, 1, 2$ with corresponding wavelengths $\sigma = \pi, 2\pi, 4\pi$. For $s = 0.5$ ($q \approx 0.21$) we see that the neighbouring solitary-wave shapes have large overlaps and sum to produce an approximate sine wave with small amplitude. This is the linear-wave regime for the periodic solution given by $s \to 0$, $q \to 1$, when nonlinear effects are weak (see section 7). As the wavelength increases, the component solitary-wave forms gradually separate and their overlaps are reduced significantly. When $s = 2$ ($q \approx 0.0019$), the solitary-wave shapes have become well separated with little overlap of their exponential tails. This is the 'large-amplitude' regime in which nonlinear effects are at their strongest and each wave crest is well approximated by a solitary wave. Indeed, letting $q \to 0$ ($s \to +\infty$) the wavelength becomes infinite, and (45) and (49) reduce to

$$u(x, t) = u_0 + \frac{p \sin \gamma}{\cosh(px + \omega t + \alpha) + \cos \gamma} \qquad 0 < \gamma < \pi \tag{54}$$

$$\omega = -(\lambda + 2u_0)p + p^2 \cot \gamma \tag{55}$$

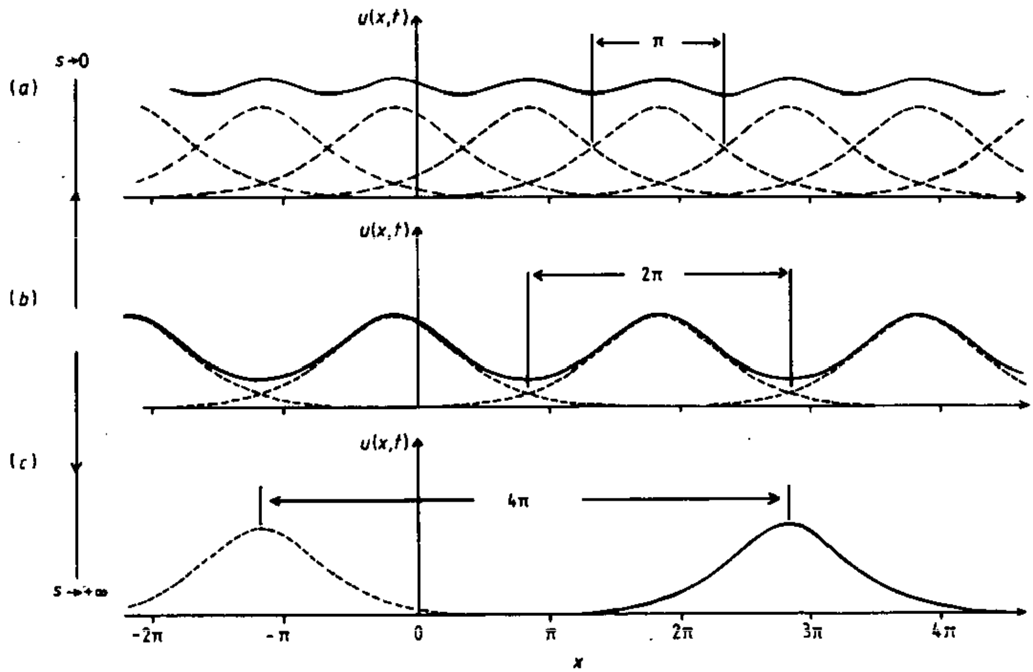

Figure 1. Generating the periodic solution of the ILW equation by the superposition of identical solitary-wave profiles, with $\gamma = \frac{1}{4}\pi$: the wave crests are broad and similar in shape to those of the cnoidal-wave solution of the KdV equation. (*a*) $s \approx 0.5$: small-amplitude linear wave regime; (*b*) $s = 1$: intermediate-amplitude wave; (*c*) $s = 2$: large-amplitude solitary-wave regime.

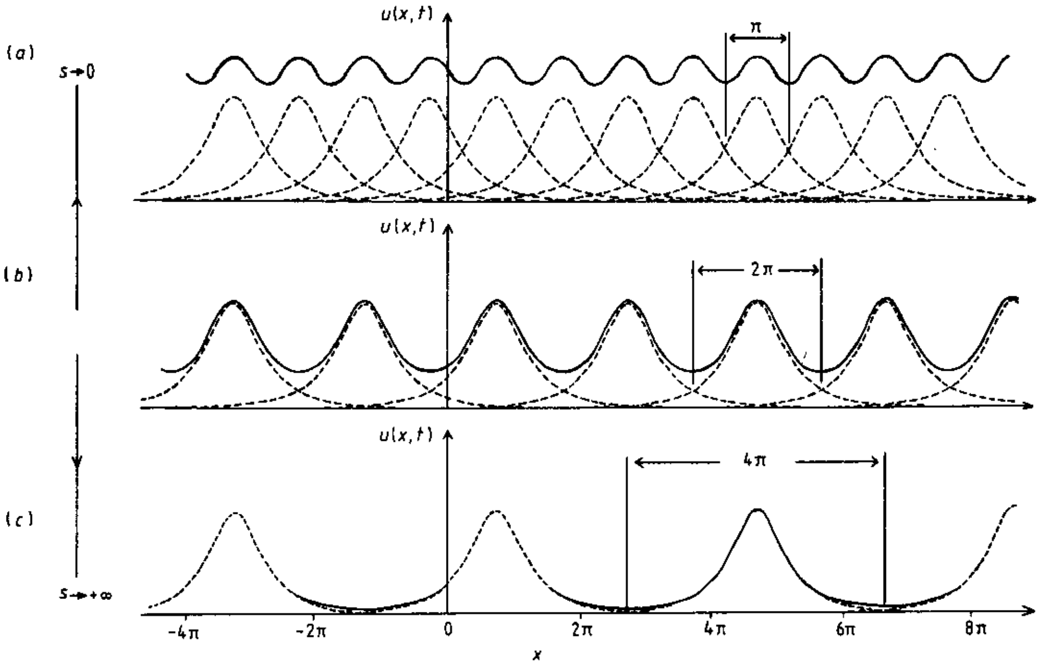

Figure 2. Periodic solution of the ILW equation with $\gamma = 3\pi/4$: the wave crests are narrow and the resulting wave is characteristic of the BO periodic wave. (*a*) $s = 0.5$; (*b*) $s = 1$; (*c*) $s = 2$.

which recovers the ILW solitary wave (with $u \to u_0$ as $|x| \to \infty$). From (33) (with $p \to \mathrm{i}p$, $\gamma \to \mathrm{i}\gamma$) we see that $B(\gamma; q) = -8p^2q^2 \sin^2 \gamma + \mathrm{O}(q^4)$, as $q \to 0$, i.e. corresponding to the solitary-wave solution (54) we must have $B = 0$, as we earlier anticipated (see section 3).

The solution (52) ($\gamma = \frac{1}{2}\pi$) divides the spectrum, $0 < \gamma < \pi$, of periodic solutions of the ILW equation into essentially two categories. When $0 < \gamma < \frac{1}{2}\pi$, the wave crests have the broad, bell-shaped character of the underlying solitary-wave form which is reminiscent of the familiar sech^2 profile of the KdV solitary wave (figure 1). This contrasts with the more sharply peaked crests when $\frac{1}{2}\pi < \gamma < \pi$, for which the underlying solitary wave has the characteristic shape of the rational (Lorentzian) solitary wave (equation (86)) of the BO equation (figure 2). This dual nature of the ILW solutions reflects the fact that the ILW equation spans the divide between the shallow-water (KdV) theory ($\gamma \to 0$) at one end, and the deep-water (BO) theory ($\gamma \to \pi$) at the other; these limits are examined in sections 9 and 10, respectively.

It is interesting to note that the solitary wave (54) is given by the term $n = 0$ in the sum (45) and may therefore be expressed in the form

$$u(x, t) = u_0 - \tfrac{1}{2}\mathrm{i}p\left\{\tanh\left(\frac{z+\mathrm{i}\gamma}{2}\right) - \tanh^*\left(\frac{z+\mathrm{i}\gamma}{2}\right)\right\}$$

where we have made use of (44) and the relationship $\tanh^*(z) = \tanh(z^*)$, the asterisk denoting the complex conjugate.

The series (45) shows that the solitary wave acts as a kind of 'template' or 'pattern' function which, when repeated at equal intervals over all space, generates the ILW periodic wave and, remarkably, does so for all values of the nome q (and not simply for values $q \approx 0$ at the solitary-wave end of the spectrum). Because of the way in which the copies of the solitary-wave shapes repeatedly overlap, Boyd [32] introduced the more suggestive term 'imbricate' to describe these series (after the adjective meaning 'to decorate with a repeating pattern like overlapping tiles') and we shall use this description in what follows.

The importance of the imbricate series (45) should not be underestimated: it is evident that it is the appropriate perturbation series for efficiently exploring the solitary-wave regime of the ILW periodic solution. As far as we are aware, the series cannot be generated by any known perturbation scheme. This contrasts with the corresponding Fourier series representation (see section 7) which is used to examine the linear-wave regime and which is obtainable by, for example, the method of multiple scales (or strained coordinates).

It is rewarding to note that, under the inverse of the transformation (4), the imbricate series (45) and wavespeed (50) together recover the periodic solution of the dimensional form of the ILW (3) as found by Miloh [20]. By virtue of (42), this solution has (using Miloh's notation) the compact analytic expression

$$u(x,t) = \mathrm{i}\frac{c_0 d}{C}\frac{\partial}{\partial x}\ln\left[\frac{\theta_2[\frac{1}{2}(\mathrm{i}k\xi+\gamma)]}{\theta_2[\frac{1}{2}(\mathrm{i}k\xi-\gamma)]}\right] \qquad \gamma = kD \qquad 0<\gamma<\pi$$

with $\xi = x - Vt$ and

$$V = c_0\left\{1+\frac{d}{2D}\left[1-\gamma\frac{\theta_1'(\gamma)}{\theta_1(\gamma)}\right]\right\}.$$

7. Fourier series: the linear-wave approximation

Whereas the imbricate series (45) converges rapidly in the solitary-wave regime $q \to 0$, this series is of little value when exploring the linear-wave domain given by $q \to 1$ ($s \to 0$). Although the series is absolutely convergent for all $0<q<1$, it is evident that the rate of convergence slows as the small-amplitude linear-wave limit $q \to 1$ is approached (see figures 1 and 2). The appropriate perturbation parameter in the latter limit is the complementary nome $q' = \exp(\mathrm{i}\pi\tau')$ where $\tau' = -1/\tau$, whence $q' = \exp(-\pi/s) \to 0$ as $q \to 1$.

To deduce the perturbation series in q' for $u(x, t)$, we make use of Jacobi's modular transformation for $\theta_2(z)$ [31]

$$\theta_2(z\,|\,\tau) = (-\mathrm{i}\tau')^{1/2}\,\mathrm{e}^{\mathrm{i}\tau' z^2/\pi}\,\theta_4(\tau' z\,|\,\tau')$$

which, when applied to (42), yields the periodic solution in the form

$$u(x,t) = u_0 - \frac{\mathrm{i}\tau'\gamma}{\pi}p + \mathrm{i}\frac{\partial}{\partial x}\ln\left[\frac{\theta_4[\frac{1}{2}\mathrm{i}\tau'(z-\mathrm{i}\gamma)|\tau']}{\theta_4[\frac{1}{2}\mathrm{i}\tau'(z+\mathrm{i}\gamma)|\tau']}\right]. \tag{56}$$

If we now use the Fourier series [31]

$$\frac{\theta_4'(z)}{\theta_4(z)} = 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin 2nz \qquad |\mathrm{Im}(z)| < \tfrac{1}{2}\pi\,\mathrm{Im}(\tau)$$

then (56) becomes

$$u(x,t) = u_0 + \frac{\gamma p}{\pi s} + \frac{4p}{s}\sum_{n=1}^{\infty}\frac{q'^n}{1-q'^{2n}}\sinh\left(\frac{n\gamma}{s}\right)\cos\left(\frac{nz}{s}\right) \qquad 0<\gamma<\pi \tag{57}$$

(the analyticity condition (41) ensuring the convergence of the series) and is the desired Fourier series representation of the periodic solution in the perturbation parameter q'.

The corresponding dispersion relation (in q') can be obtained by applying Jacobi's transformation for θ_1 to (38) which results in

$$\omega=-\left[\lambda+2\left(u_0+\frac{\gamma p}{\pi s}\right)\right]p+\mathrm{i}\frac{p^2}{s}\frac{\theta_1'(\mathrm{i}\gamma/s,q')}{\theta_1(\mathrm{i}\gamma/s,q')}.$$

If we now use the expansion (48) in the last result, we obtain the series

$$\omega=-\left[\lambda+2\left(u_0+\frac{\gamma p}{\pi s}\right)\right]p+\frac{p^2}{s}\left\{\coth\left(\frac{\gamma}{s}\right)-4\sinh\left(\frac{2\gamma}{s}\right)\sum_{n=1}^{\infty}\frac{q'^{2n}}{1-2q'^{2n}\cosh(2\gamma/s)+q'^{4n}}\right\}. \tag{58}$$

Letting $q'\to 0$ $(s\to 0)$ in (57) and (58), we deduce the 'small-amplitude' linear-wave approximation for the periodic ILW solution

$$u(x,t)\sim u_0+\frac{\gamma p}{\pi s}+\frac{2p}{s}\mathrm{e}^{-(\pi-\gamma)/s}\cos\left(\frac{z}{s}\right) \qquad 0<\gamma<\pi \tag{59}$$

$$\omega=-\left[\lambda+2\left(u_0+\frac{\gamma p}{\pi s}\right)\right]p+\frac{p^2}{s}\coth\left(\frac{\gamma}{s}\right)+\mathrm{O}(s^{-1}q'^{2(1-\gamma/\pi)}) \tag{60}$$

which we recognize as the infinitesimal Stokes solution with the 'correct' linear-wave speed

$$c=2\left(u_0+\frac{\gamma p}{\pi s}\right)+\lambda\left[1-\frac{\gamma}{s}\coth\left(\frac{\gamma}{s}\right)\right].$$

8. Intermediate-amplitude waves

We have seen that, when the amplitude is small (i.e. nonlinear effects are weak), the periodic solution of the ILW equation is approximately a sine wave, whereas for 'large' amplitudes (when nonlinear effects are at their strongest) it is well approximated by a solitary wave. We might reasonably enquire as to the nature of the periodic wave in the regime of intermediate amplitudes and moderate nonlinearity.

Boyd [33] investigated this for the periodic (cnoidal-wave) solution of the KdV equation and showed (numerically) that, in the regime of intermediate amplitudes, the cnoidal wave may be regarded as either a sine wave or solitary wave to a good approximation. We will show that this is also true of the more general ILW periodic wave, and, in doing so, obtain analytic expressions for the errors involved.

To proceed, we shall find it convenient to normalize the ILW periodic solution in the conventional way by setting the ambient (or mean) level of the wave

$$\langle u\rangle=\frac{1}{\sigma}\oint u(x,t)\,\mathrm{d}x \tag{61}$$

to zero. (The integration in (61) is over a single wavelength σ.) This requires that the constant term in the Fourier series representation (57) is zero, and we therefore set $u_0=-\gamma p/\pi s$.

We take as our linear-wave approximation the first term in the Fourier series (57)

$$W(z;\,\gamma,q')=\frac{4p}{s}\cdot\frac{q'}{1-q'^2}\sinh\left(\frac{\gamma}{s}\right)\cos\left(\frac{z}{s}\right) \tag{62}$$

which we anticipate will be a slightly better approximation than the Stokes solution (59).

To see how well $W(z)$ approximates to $u(x,t)$, we consider the error function (given by subtracting (57) and (62))

$$E_1(z;\,\gamma,q')=u(x,t)-W(z)=\frac{4p}{s}\sum_{n=2}^{\infty}\frac{q'^n}{1-q'^{2n}}\sinh\left(\frac{n\gamma}{s}\right)\cos\left(\frac{nz}{s}\right) \tag{63}$$

where the subscript indicates a first-order approximation. It is evident that $E_1(z)$ is periodic ($2\pi s$) and even, and so we need only consider the half-period $0\leqslant z\leqslant \pi s$. From (63) we easily see that the maximum error occurs at $z=0$, whence we have that

$$|E_1(z)|\leqslant\frac{4p}{s}\sum_{n=2}^{\infty}\frac{q'^n}{1-q'^{2n}}\sinh\left(\frac{n\gamma}{s}\right)<\frac{4p}{s(1-q'^4)}\sum_{n=2}^{\infty}q'^n\sinh\left(\frac{n\gamma}{s}\right) \tag{64}$$

where we have used $0<q'<1$ to obtain this last inequality.

The last series in (64) can be majorized by recalling that $q'=\exp(-\pi/s)$, so that we obtain a global error bound

$$|E_1(z)|<e_1(\gamma,q')=\frac{2p}{s}\cdot\frac{q'^{2(1-\gamma/\pi)}}{(1-q'^4)[1-q'^{(1-\gamma/\pi)}]}\qquad 0<\gamma<\pi. \tag{65}$$

We notice that $e_1\to 0$ as $q'\to 0$, as expected, since this gives the small-amplitude linear-wave limit.

Because the linear-wave and solitary-wave approximations are given by $q'\to 0$ and $q\to 0$, respectively, the 'worst-case' scenario for intermediate amplitudes occurs when $q'=q=\exp(-\pi)$, i.e. when $s=1$. Moreover, as the error bound is proportional to p, we consider the case $p=1$. Then, for the 'typical' periodic wave with $\gamma=\frac{1}{2}\pi$, we find that, in the 'worst-case' scenario, the error bound (65) is

$$e_1=2\,\mathrm{e}^{-\pi}/\{(1-\mathrm{e}^{-4\pi})(1-\mathrm{e}^{-\pi/2})\}=0.109$$

which gives a relative error of 21.4% at $z=0$. This shows that there is reasonably good agreement between the linear wave $W(z)$ and periodic solution $u(x,t)$ even in the regime of intermediate amplitudes (figure 3). If we now put $s=0.5$ ($q'=\exp(-2\pi)\approx 0.0019$) we find that the error estimate is significantly reduced to $e_1=0.0078$ (a relative error at $z=0$ of only 4.3%) and the graphs of the periodic solution and approximating linear wave are virtually indistinguishable. We conclude that the 'typical' ILW periodic wave can be legitimately regarded as a sine wave even in the regime of moderate nonlinearity. This should not surprise us since it reflects the weakly nonlinear character of the ILW equation.

We note, however, that the error bound e_1 is a function of γ which, as we have seen, characterizes the shape of the periodic wave. Thus, for a fixed p and q', (65) shows that the error is reduced as we approach the KdV end of the ILW periodic spectrum (i.e. $\gamma\to 0$), and increases at the BO end ($\gamma\to\pi$). In other words, the intermediate-amplitude periodic solution of the ILW equation is better approximated by a sine wave for 'KdV-like' waves than for 'BO-like' waves.

Now let us approximate the periodic solution on $[-\pi s,\pi s]$ by the (normalized) solitary wave centred at $z=0$ and given by (54) as

$$V(z;\,\gamma,q)=-\frac{p\gamma}{\pi s}+\frac{p\sin\gamma}{\cosh z+\cos\gamma}. \tag{66}$$

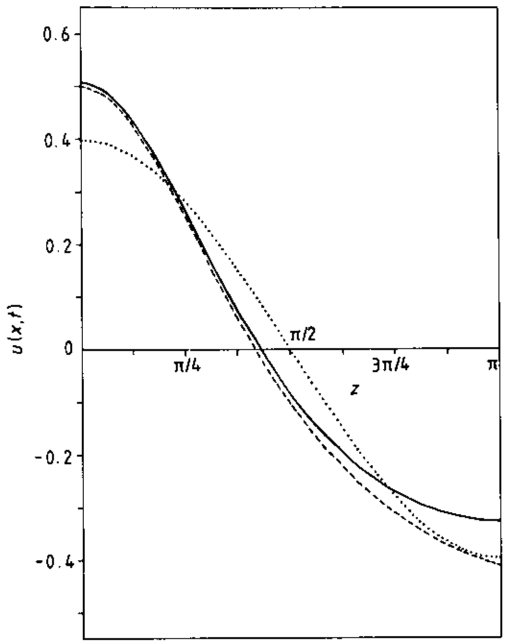

Figure 3. The (normalized) intermediate-amplitude ILW periodic wave with $\gamma = \frac{1}{2}\pi$ and $p=1$ (full curve) is compared with the approximating linear wave $W(z) = \text{sech}(\frac{1}{2}\pi)\cos z$ (dotted curve) and solitary wave $V(z) = -0.5 + \text{sech}\, z$ (broken curve) in the 'worst-case' scenario $q = q' = \exp(-\pi)$.

Then, (45) and (66) yield the error function

$$\mathscr{E}_1(z;\gamma,q) = u(x,t) - V(z) = p \sum_{n=-\infty}^{\infty}{}' \frac{\sin\gamma}{\cosh(z - 2n\pi s) + \cos\gamma} > 0 \tag{67}$$

where the summation excludes the term $n = 0$. Again it is sufficient to consider $0 \leqslant z \leqslant \pi s$ since $\mathscr{E}_1(z)$ is even.

It is evident from the superposition (45) that the maximum error now occurs at $z = \pi s$ (because the largest contribution to $\mathscr{E}_1$ on $[0, \pi s]$ is from the solitary-wave profile immediately to the right of $V(z)$ and centred at $z = 2\pi s$). Hence, substituting $z = \pi s$ into equation (67) and making use of the relation $q = \exp(-\pi s)$, we eventually find that

$$\mathscr{E}_1(z;\gamma,q) \leqslant 2p\sin\gamma \left[\frac{q}{1+2q\cos\gamma+q^2} + 2\sum_{n=2}^{\infty} \frac{q^{2n-1}}{1+2q^{2n-1}\cos\gamma + q^{2(2n-1)}}\right]. \tag{68}$$

If we now recall that $0 < \gamma < \pi$ and $0 < q < 1$, then (68) finally yields an error bound

$$\mathscr{E}_1(z) < \varepsilon_1(\gamma, q) = \frac{2pq\sin\gamma}{(1-q)^2}\left[1 + \frac{2q^2}{1-q^3}\right]. \tag{69}$$

In the solitary-wave limit $q \to 0$, we see that $\varepsilon_1 \to 0$, as expected.

Once again, the error bound is proportional to p and it suffices to consider the case $p = 1$. The global error is a maximum for the 'typical' periodic wave, $\gamma = \frac{1}{2}\pi$: for the 'worst-case' scenario $s = 1$ $(q = q' = \exp(-\pi))$, we obtain $\varepsilon_1 = 0.095$ which gives a relative error at $z = \pi s$ of approximately 29%. This rather large (but not surprising) error at $z = \pi s$ belies the fact that the solitary wave $V(z)$ is a good approximation for

$u(x, t)$ over the greater part of the period $[-\pi s, \pi s]$ (as figure 3 clearly shows). If we put $s=2$ ($q=\exp(-2\pi)$) in (69), then the error is significantly reduced to 0.0037 (a relative error of only 1.5% at $z=\pi s$) in which case the periodic wave and approximating solitary wave are graphically indistinguishable. Thus, to a good approximation, the ILW periodic wave may be regarded as a solitary wave even in the regime of intermediate amplitudes.

We conclude that the periodic solution of the ILW equation exhibits a large overlap between the linear- and solitary-wave regimes; this duality reflects the weak nonlinearity of the ILW equation itself. Put another way, periodic waves with intermediate amplitude may be regarded as either a solitary wave or linear wave to a good approximation.

9. The KdV (shallow-water) limit

The shallow-water limit for the ILW (3) is given by $D\to 0$, and equation (5) shows that this is equivalent to $\lambda\to\infty$. Consider the dimensionless form of the ILW equation given by (8) and (9), and note that

$$\bar{c}(k)=-\frac{1}{3\lambda}k^2+\mathrm{O}(\lambda^{-3}) \qquad \text{as } \lambda\to\infty. \tag{70}$$

If we now introduce the coordinate transformation

$$x=\tilde{x}/\sqrt{\lambda} \qquad t=3\tilde{t}/\sqrt{\lambda} \tag{71}$$

and use equation (70), then, in the shallow-water limit $\lambda\to\infty$, the ILW (8)–(9) reduces to the KdV equation

$$u_{\tilde{t}}+6uu_{\tilde{x}}+u_{\tilde{x}\tilde{x}\tilde{x}}=0. \tag{72}$$

To deduce the corresponding limit for the (real) periodic solution of the ILW equation we define new parameters

$$\tilde{p}=p/\sqrt{\lambda} \qquad \tilde{\omega}=3\omega/\sqrt{\lambda} \qquad \tilde{\alpha}=\alpha \tag{73}$$

where $\tilde{p}$, $\tilde{\alpha}$ are real and finite as $\lambda\to\infty$. We note that $\gamma=p/\lambda=\tilde{p}/\sqrt{\lambda}\to 0$ as $\lambda\to\infty$.

Substituting (71) and (73) into (45) and letting $\lambda\to\infty$, we deduce the periodic solution of the KdV equation (72)

$$u(x,t)=u_0+\tfrac{1}{2}\tilde{p}^2\sum_{n=1}^{\infty}\operatorname{sech}^2\tfrac{1}{2}(z-2n\pi s) \qquad z=\tilde{p}\tilde{x}+\tilde{\omega}\tilde{t}+\tilde{\alpha} \tag{74}$$

with wavelength $\sigma=2\pi s/\tilde{p}$.

Of course, (74) is nothing more than the cnoidal-wave solution of the KdV equation expressed as a superposition of identical sech^2 solitary-wave profiles. This result was the first reported example of the nonlinear superposition principle for periodic solutions of evolution equations; it was first obtained by Toda [21] and later by Korpel and Banerjee [22], Whitham [23] and Boyd [34] using other methods. The cnoidal wave (74) has a representation in terms of theta functions which can be deduced from the corresponding form for ILW periodic solution (42): using the same limiting procedure we find that

$$u(\tilde{x},\tilde{t})=u_0+2\partial_x^2\ln\theta_2(\tfrac{1}{2}\mathrm{i}z,q).$$

This is precisely the form of the cnoidal-wave solution obtained by Nakamura [35] and Parker [30] using the bilinear transformation method.

Substituting (73) into (38), and using the series (48), we eventually derive the dispersion relation corresponding to (74)

$$\tilde{\omega} = -6u_0\tilde{p} + \tilde{p}^3 \frac{\theta_1'''(0, q)}{\theta_1'(0, q)} \tag{75}$$

where we have made use of the standard result [31]

$$\frac{\theta_1'''(0, q)}{\theta_1'(0, q)} = -1 + 24 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2}.$$

The expression (75) for the dispersion relation corresponding to the cnoidal wave of the KdV equation has been obtained by the present author [30] using other (more direct) means and would seem to be new.

Letting $\sigma \to \infty$ (i.e. $q \to 0$, $s \to \infty$) in equations (74) and (75), we recover the classical sech^2 solitary-wave solution of the KdV equation

$$u(\tilde{x}, \tilde{t}) = u_0 + \tfrac{1}{2}\tilde{p}^2 \,\text{sech}^2 \tfrac{1}{2}(\tilde{p}\tilde{x} + \tilde{\omega}\tilde{t} + \tilde{\alpha}) \qquad \tilde{\omega} = -6u_0\tilde{p} - \tilde{p}^3. \tag{76}$$

Taking the KdV limit in the integration constant $B(\gamma; q)$ (equation (32) with $p \to \mathrm{i}p$, $\gamma \to \mathrm{i}\gamma$) it can be shown that

$$B = -8\tilde{p}^4 \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{(1-q^{2n})^2} = \tfrac{1}{8}\tilde{p}^4 \lim_{z \to 0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{\theta_1''}{\theta_1}\right] \tag{77}$$

and therefore $B \neq 0$ for the periodic problem; however, we see that $B = 0$ in the solitary-wave limit $q \to 0$.

10. The BO (deep-water) limit

The deep-water limit is obtained by letting $D \to \infty$ and leads to $\lambda \to 0$, in which case equation (9) gives $\bar{c}(k) \sim -|k|$. Substituting for $\bar{c}(k)$ into the ILW (8) now yields the BO equation

$$u_t + 2uu_x + \mathscr{H}[u]_{xx} = 0 \tag{78}$$

where $\mathscr{H}[f]$ is the Hilbert transform

$$\mathscr{H}[f(x)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x'.$$

To find the limiting form of the (real) ILW periodic solution (45) in the deep-water limit, we first introduce the arbitrary (finite) parameter $\tilde{p} > 0$ such that

$$p = \pi\lambda(1 - \lambda/\tilde{p}). \tag{79}$$

It follows that

$$\gamma = p/\lambda = \pi(1 - \lambda/\tilde{p}) \tag{80}$$

which remains finite in the deep-water limit $\lambda \to 0$ with $\gamma \to \pi$.

To proceed, we now define the finite parameter $\tilde{s}$ by $\tilde{s}/\tilde{p} = s/p$, for all λ, whereupon the wavelength of the periodic solution (45)

$$\sigma = 2\pi s/p = 2\pi\tilde{s}/\tilde{p} \tag{81}$$

is independent of λ. We note that p and s are both $\mathrm{O}(\lambda)$ as $\lambda \to 0$.

Then, using equations (79)-(81), we find that the periodic wavespeed (50) has the limiting form

$$c=2u_0+\tilde{p}\left\{1+2\sum_{n=1}^{\infty}\left[1+\left(\frac{n\sigma\tilde{p}}{2}\right)^2\right]^{-1}\right\}+\mathrm{O}(\lambda) \qquad \text{as } \lambda\to 0.$$

This last series may be summed by using the Mittag-Leffler expansion

$$\pi z\coth \pi z=1+2\sum_{n=1}^{\infty}[1+(n/z)^2]^{-1}$$

which, with $z=2/\sigma\tilde{p}$, yields

$$c=\tilde{c}+\mathrm{O}(\lambda) \qquad \tilde{c}=2u_0+\frac{2\pi}{\sigma}\coth\left(\frac{2\pi}{\sigma\tilde{p}}\right) \qquad \text{as } \lambda\to 0. \tag{82}$$

If we now put

$$\tilde{\omega}=-\tilde{p}\tilde{c} \qquad \text{and} \qquad \alpha=p\tilde{\alpha}/\tilde{p} \tag{83}$$

(with $\tilde{\alpha}$ finite), then the phase variable $z=px+\omega t+\alpha$ becomes

$$z=\frac{\pi\lambda}{\tilde{p}}(\tilde{p}x+\tilde{\omega}t+\tilde{\alpha})+\mathrm{O}(\lambda^2) \qquad \text{as } \lambda\to 0. \tag{84}$$

Substituting (84) into (45), and proceeding to the limit as $\lambda\to 0$, finally reveals

$$u(x,t)=u_0+2\tilde{p}\sum_{n=-\infty}^{\infty}\frac{1}{1+(\tilde{z}-2n\pi\tilde{s})^2} \qquad \tilde{z}=\tilde{p}x+\tilde{\omega}t+\tilde{\alpha} \tag{85}$$

a periodic solution of the BO equation (78) with wavelength given by (81).

Applying the same limiting procedure to the integration constant B, we can show that

$$B\sim\frac{\tilde{p}\lambda}{\tilde{s}}\left[1-\frac{1}{\tilde{s}}\coth\left(\frac{1}{\tilde{s}}\right)\right] \qquad \text{as } \lambda\to 0$$

and it follows that the integration constant B must vanish in the BO limit for (85) to be a solution of the Benjamin-Ono equation (cf the non-zero KdV limit (77)).

Letting the wavelength $\sigma\to\infty$ (i.e. $\tilde{s}\to\infty$), equations (82), (83) and (85) yield

$$u(x,t)=u_0+\frac{2\tilde{p}}{1+\tilde{z}^2} \qquad \tilde{z}=\tilde{p}x+\tilde{\omega}t+\tilde{\alpha} \tag{86a}$$

with

$$\tilde{\omega}=-2u_0\tilde{p}-\tilde{p}^2 \tag{86b}$$

which is the algebraic (Lorentzian) solitary wave of the BO equation first derived by Benjamin [13]. Equations (85) and (86) now show that the periodic solution of the BO equation can be represented by an infinite sum of identical Lorentzian solitary-wave profiles; this is yet another example of the nonlinear superposition principle. The result for the BO equation seems to have been demonstrated for the first time by Zaitsev [25] using a Fourier series approach; it has been reported more recently by Miloh and Tulin [24] who used a method based on contour integration. We observe that, yet again, the periodic-wave speed $\tilde{c}$ (equation (82)) differs from the speed of the solitary wave

$$\tilde{c}_s=2u_0+\tilde{p}$$

(given by equation (86*b*)), except in the solitary-wave limit $\sigma\to\infty$.

The imbricate series (85) can be summed by using the identity

$$\sum_{n=-\infty}^{\infty} \frac{x}{x^2+(y-n\pi)^2} = \frac{\sinh 2x}{\cosh 2x - \cos 2y} \qquad (x, y) \neq (0, n\pi) \tag{87}$$

which can be deduced from the Mittag-Leffler expansion

$$\coth z = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{2z}{z^2+m^2\pi^2}.$$

Putting $x = 1/2\tilde{s}$ and $y = \tilde{z}/2\tilde{s}$ in (87) results in

$$\sum_{n=-\infty}^{\infty} \frac{2\tilde{s}}{1+(\tilde{z}-2n\pi\tilde{s})^2} = \frac{\sinh(1/\tilde{s})}{\cosh(1/\tilde{s}) - \cos(\tilde{z}/\tilde{s})}. \tag{88}$$

Substituting (88) into (85) gives

$$u(x, t) = u_0 + \frac{(\tilde{p}/\tilde{s})\sinh(1/\tilde{s})}{\cosh(1/\tilde{s}) - \cos(\tilde{z}/\tilde{s})} \tag{89}$$

which we recognize as the periodic solution of the BO equation found by Benjamin [13] and Ono [14].

The Fourier series for the periodic solution (89) can be deduced from that for the ILW periodic wave, equation (57); proceeding to the BO limit we obtain

$$u(x, t) = u_0 + \frac{\tilde{p}}{\tilde{s}}\left[1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-n/\tilde{s}} \cos(n\tilde{z}/\tilde{s})\right]. \tag{90}$$

It is straightforward to sum this last series to recover the right-hand side of (89).

The periodic solution (89) has its dispersion relation given by equations (82) and (83) as

$$\tilde{\omega} = -2u_0\tilde{p} - \frac{\tilde{p}^2}{\tilde{s}} \coth\left(\frac{1}{\tilde{s}}\right) \tag{91}$$

and then, letting $\tilde{s} \to 0$ (i.e. $\sigma \to 0$) in (90) and (91), we deduce the small-amplitude linear-wave approximation for the BO periodic wave

$$u(x, t) \sim u_0 + \frac{\tilde{p}}{\tilde{s}} + 2\frac{\tilde{p}}{\tilde{s}} \mathrm{e}^{-1/\tilde{s}} \cos\left(\frac{\tilde{z}}{\tilde{s}}\right)$$

with

$$\tilde{\omega} \sim -2u_0\tilde{p} - \tilde{p}^2/\tilde{s}.$$

This, of course, is the infinitesimal Stokes solution for the BO equation which can be confirmed by taking the limit $\lambda \to 0$ in (59) and (60) (noting, in particular, that $(\pi - \gamma)/s \to 1/\tilde{s}$).

By letting $\lambda \to 0$, $B \to 0$ in (12), we see that the bilinear form of the BO equation is

$$[\mathrm{i}D_t + 2\mathrm{i}u_0D_x - D_x^2]f_+ \cdot f_- = 0.$$

The corresponding bilinear form of the BO periodic solution (89) can then be obtained from (42) in the BO limit. However, it is more straightforward to use the expression in (89) which, together with the identity

$$\coth A - \coth B = \frac{2\sinh(A-B)}{\cosh(A-B) - \cosh(A+B)}$$

yields

$$u(x,t)=u_0+\mathrm{i}\partial_x \ln(f_+/f_-)$$

where

$$f_+(x,t)=1-\exp[(\mathrm{i}\tilde{z}-1)/\tilde{s}]$$

$$f_-(x,t)=1-\exp[(\mathrm{i}\tilde{z}+1)/\tilde{s}].$$

These results were first reported by Satsuma and Ishimori [36] (for the case $u_0=0$).

11. Further periodic solutions of the ILW equation

Joseph and Egri [2] and Chen and Lee [17] conjectured that a real periodic solution of the ILW (6) can be derived by letting $p\to \mathrm{i}p$, $\omega\to\mathrm{i}\omega$, etc, in the solitary wave (54)-(55). This somewhat *ad hoc* procedure leads to the solution

$$u(x,t)=u_0-\frac{p\sinh\gamma}{\cos z+\cosh\gamma}\qquad z=px+\omega t+\alpha \tag{92}$$

$$\omega=-(\lambda+2u_0)p+p^2\coth\gamma \tag{93}$$

with α real.

However, Ablowitz *et al* [27] claim that (92) does not, in fact, solve (6) and give the 'correct' solution as

$$u(x,t)=u_0-\frac{p\sinh\gamma}{\cos(z+\mathrm{i}\phi)+\cosh\gamma} \tag{94}$$

where $\phi\neq 0$ and z are real, i.e. the phaseshift $\alpha+\mathrm{i}\phi$ is complex definite. This solution is complex-valued. These authors do not actually derive the solution (94), but merely state it as a fact and then verify its validity. Moreover, they do not indicate how the solution is to be obtained.

To obtain the 'corrected' (complex) version (94) of the proposed (real) periodic solution (92), we shall make use of the solution given by equation (36). Transforming $\alpha\to\alpha+\mathrm{i}\pi s$ (i.e. $z\to z+\pi\tau$), and using the equivalent expression (30), yields the periodic solution of the ILW (6) in the form

$$u(x,t)=u_0+\mathrm{i}\partial_x \ln\left[\frac{\theta_2[\frac{1}{2}(z-\mathrm{i}\gamma)]}{\theta_2[\frac{1}{2}(z+\mathrm{i}\gamma)]}\right] \tag{95}$$

where we have again made use of the quasiperiodicity relation $\theta_2(z)=\mu\theta_3(z+\frac{1}{2}\pi\tau)$ (see section 6). The dispersion relation for (95) is unaltered and is given by equation (31). However, the analyticity condition (37) is transformed to

$$-2\pi s+\gamma<\mathrm{Im}(\alpha)<-\gamma\qquad 0<\gamma<\pi s. \tag{96}$$

Alternatively, we can let $\alpha\to\alpha-\mathrm{i}\pi s$ (i.e. $z\to z-\pi\tau$) in (36) which also leads to (95) (the solution is periodic in z with period $2\mathrm{i}\pi s$) but the domain of validity is now

$$\gamma<\mathrm{Im}(\alpha)<2\pi s-\gamma\qquad 0<\gamma<\pi s. \tag{97}$$

We observe that the periodic solution (95) is complex-valued since $\mathrm{Im}(\alpha)\neq 0$.

Substituting (43) into (95) we easily deduce the imbricate series representation

$$u(x,t)=u_0-p\sum_{n=-\infty}^{\infty}\frac{\sinh\gamma}{\cos(z-2in\pi s)+\cosh\gamma}\qquad 0<\gamma<\pi s$$

where the corresponding dispersion relation is given by (31) and (48) as

$$\omega=-(\lambda+2u_0)p+p^2\coth\gamma-4p^2\sinh\gamma\sum_{n=1}^{\infty}\frac{q^{2n}}{1-2q^{2n}\cosh 2\gamma+q^{4n}}\qquad q=e^{-\pi s}.$$

Proceeding to the limit $q\to 0$ (i.e. $s\to\infty$), these last two results give

$$u(x,t)=u_0-\frac{p\sinh\gamma}{\cos z+\cosh\gamma}\qquad z=px+\omega t+\alpha \tag{98a}$$

$$\omega=-(\lambda+2u_0)p+p^2\coth\gamma. \tag{98b}$$

If we rewrite the phaseshift as $\alpha+i\phi$ (α, ϕ real) and let $s\to\infty$ in (96) and (97), then we see that the periodic solution (98) has its domain of analyticity defined by $\phi<-\gamma$ or $\phi>\gamma$ ($\gamma>0$). Thus, the periodic solution (98) recovers the complex-valued solution (94) given by Albowitz *et al* [27] and is valid in the entire complex plane except in the strip $|\mathrm{Im}(z)|\leq\gamma$, $\gamma>0$. This solution can never be real-valued and confirms that the (real) solution (92) proposed by Joseph and Egri [2] and Chen and Lee [17] is incorrect.

Even though we have taken $p>0$ in (98), the solutions in the two subregions $\phi<-\gamma$ and $\phi>\gamma$ are not independent (since the solution is parity invariant and defined on $-\infty<x<+\infty$). We therefore need only consider the solution in the upper domain $\mathrm{Im}(z)=\phi>\gamma$. In this case, its Fourier series representation is

$$u(x,t)=u_0+2p\sum_{n=-\infty}^{\infty}(-1)^n\, e^{in(z+i\phi)}\sinh(n\gamma)$$

which converges for $\phi>\gamma$ ($\gamma>0$).

The advantage of our approach is now apparent: by starting with a complex periodic solution of the ILW (6) we are able to deduce the complex periodic solution, given by Ablowitz *et al*, as a particular limiting case.

That (98) solves the ILW (6) (subject to the analyticity requirement $\phi>\gamma$) can readily be seen by writing the solution in its bilinear form

$$u(x,t)=u_0+i\partial_x\ln[(1+e^{iz+\gamma})/(1+e^{iz-\gamma})]. \tag{99}$$

If we compare (99) with (11), we see that $f(x,t)=1+\exp(iz)$ which, upon substitution into (12), will result in a solution of the ILW bilinear equation if

$$-2e^{iz}\{[\omega+p(\lambda+2u_0)]\sinh\gamma-p^2\cosh\gamma\}+B(1+2e^{iz}\cosh\gamma+2e^{2iz})=0. \tag{100}$$

But, using equation (33), one gets $B=\lim_{q\to 0}B(\gamma;q)=0$ and so (100) is satisfied provided ω is given by (98b), which is the desired dispersion relation.

It is interesting to note that the complex ILW periodic solution (98) has the same analytic form as the real BO wave (89) (modulo a phaseshift); however, we emphasize their different wavespeeds (cf equations (91) and (98b)) and the functional dependence $\gamma=p/\lambda$ in the case of the ILW solution.

In the deep-water limit, $\lambda\to 0$, the ILW periodic solution (98) reduces to the trivial solution u_0 of the BO equation (recall that $p=O(\lambda)$ and $\gamma\to\pi$), whereas the shallow-water limit, $\lambda\to\infty$, yields (in the notation of section 9)

$$u(\tilde{x},\tilde{t})=u_0-\tfrac{1}{2}\tilde{p}^2\sec^2\tfrac{1}{2}(\tilde{p}\tilde{x}+\tilde{\omega}\tilde{t}+\tilde{\alpha}) \tag{101a}$$

a singular solution of the KdV equation with dispersion relation

$$\tilde{\omega} = -6u_0\tilde{p} + \tilde{p}^3. \tag{101b}$$

This last solution is valid throughout the complex $\tilde{\alpha}$-plane since $\gamma \to 0$ in the analyticity condition $\mathrm{Im}(\tilde{\alpha}) = \phi > \gamma$ (and its reflection $\phi < -\gamma$). It is perhaps somewhat ironical that, in the KdV case, we could have obtained the limiting solution (101) directly from the classical sech^2 solitary wave (76) by using the *ad hoc* procedure of Joseph, Chen and others and simply replace $\tilde{p}$ by $i\tilde{p}$, etc.

12. Summary and conclusions

The ILW equation (6) has been solved for stationary periodic solutions using the bilinear transformation method following the procedure reported by Nakamura and Matsuno [26]. The resulting solution (30), expressed in terms of theta functions, is complexed-valued and has a natural parametrization in the nome q. A new expression for the dispersion relation, equation (31), has been obtained. The important class of real solutions, which correspond to physical waves, has been identified and found to agree with the periodic solutions of the dimensional ILW (3) reported by Miloh [20]. In the shallow-water limit, the ILW periodic solution reduces to the well known cnoidal wave of the KdV equation and a new expression for its dispersion relation, (75), has been deduced. The opposite, deep-water, limit yields the periodic solution of the BO equation first found by Benjamin.

The expression (45) shows that the ILW periodic solution can be represented as an 'imbricate' series of equally spaced identical solitary-wave profiles. As the wavelength σ increases, the overlaps between adjacent profiles are reduced until, in the limit $\sigma \to \infty$, the periodic solution reduces to the ILW solitary wave (54). In the limit $\sigma \to 0$, the solution is approximately a small-amplitude sinusoidal wave (the infinitesimal Stokes solution (59)). Furthermore, it has been shown that, for intermediate amplitude and moderate linearity, the ILW periodic solution can be approximated by a sine wave or solitary wave.

The controversy surrounding the (real) periodic solution proposed by Joseph and Egri [2] and Chen and Lee [17] and the alternative (complex) periodic solution given by Ablowitz *et al* [27] has been examined, and the latter solution has been shown to be correct by deriving it as the limit ($q \to 0$) of the general complex periodic solution (36).

Benjamin [13] showed that the periodic solution, (89), of the BO equation has waves which are sharper at the crests than at the troughs. As we might expect, this feature is shared by the more general ILW periodic wave (45). For, if $\bar{u} = (u_{\max} + u_{\min})/2$ denotes the elevation midway between a crest and trough, then it is straightforward to show that

$$\bar{u} = \langle u \rangle + \frac{2p}{s} \sum_{n=1}^{\infty} \frac{\sinh(2n\gamma/s)}{\sinh(2n\pi/s)}$$

where $\langle u \rangle$ is the ambient level of the wave as defined in (61). It follows immediately that $\bar{u}$ is greater than $\langle u \rangle$, indicating that the waves are narrower at the crests than at the troughs, i.e. the waves are peaked upwards.

The property, whereby the ILW periodic solution can be represented as an infinite sum of solitary-wave profiles (equation (45)), is one that is exhibited by a number of other nonlinear wave equations, notably the related KdV and BO equations. However, there has been a tendency in much of the literature to misinterpret this remarkable property by saying that the periodic solution is an exact superposition of 'solitons' (though, to be fair, this misleading view may well have more to do with a somewhat loose usage of the word 'soliton', than a genuine misunderstanding on the part of some authors). Nevertheless, the fact remains that the component solitary-wave functions are not, in general, solutions of the underlying nonlinear equation. This is because each solitary-wave profile in the sum has the same speed as the parent periodic wave which is, in general, different from the speed of the solitary wave proper. Thus, there is no genuine superposition of solutions in the accepted sense of linear theory, but only a superposing of the shapes of solitary waves, the latter acting as a kind of pattern or template.

However, Zaitsev [25] remarks that, for those equations which are Galilean invariant, the arbitrary level (u_0) may be chosen so that the periodic and solitary-wave speeds are equal. He maintains that, for this particular choice of u_0, the imbricate series is a genuine superposition of solitary waves. But this assertion is, in fact, incorrect because the required value of u_0 is invariably non-zero (except in very special circumstances: see e.g. the solution (52) when $\gamma=\frac{1}{2}\pi$). In this case, we do have a linear superposition of solutions since the non-zero constant u_0 is, in general, a trivial solution of the nonlinear wave equation, but it is not, of course, a solitary wave!

There is a divergence of opinion as to what kind of superposition principle and dynamical interaction is at work here. Miloh [20], for example, talks about a '*linear* superposition of solitons' (his emphasis), whereas Whitham [23] states that 'the representation ... may be viewed as another instance of the "clean interaction" of solitons, in that they are superposed but retain their identity and do not destroy each other under nonlinear coupling'. Toda [21], on the other hand, remarks that 'these solitons are mutually interacting, not independent of each other, and hence their speed is not given by the formula ... for a soliton, but is given by the dispersion relation ... of the [periodic] wave'.

It is evident that, in all the cases mentioned here, a very special nonlinear superposition principle is involved. Certainly, the solitary-wave profiles which generate the periodic wave are added linearly; but no linear superposition principle in the accepted sense obtains because we are not combining solitary-wave solutions as such. Neither can it be said, as Toda maintains, that the waveforms are 'mutually interacting'; after all, in the stationary coordinate frame, the solitary-wave shapes just sit there and so there is clearly no dynamical interaction whatsoever. Whitham's suggestion, supported by Miloh [20] and Miloh and Tulin [24], that the superposition property is somehow related to the 'clean interaction' property of colliding solitons, is unsatisfactory if only for the simple reason that the imbricated wave profiles retain their identity for all time which is certainly not the case for multisoliton interactions in which the individual 'solitons' regain their solitary-wave identities only after an infinite time!

Rather, it would seem more plausible to regard this nonlinear superposition principle as merely a representational one, whereby the component solitary-wave profiles in the imbricate series are simply added together with no implied dynamical interaction. It would appear that the nonlinear character resides solely in the difference between the periodic-wave speed and that of the would-be constituent solitary waves.

The nonlinear superposition principle, as outlined here for spatially periodic solutions (in one spatial dimension), has been shown to have wide application in the theory of nonlinear wave equations [20–25, 34]. However, the extent to which the principle is valid remains an open question. Certainly, it would appear that those equations which possess hyperbolic or rational solitary waves also have spatially periodic solutions which can be represented by imbricate series: to date, no exceptions are known. These periodic solutions can all be expressed in terms of elliptic functions. However, Boyd [37] has shown (numerically) that the hypercnoidal† waves of the quartic KdV equation [39] cannot be represented by an exact superposition of solitary-wave profiles (although the imbricate series is a very good approximation to the hypercnoidal wave even when there is large overlapping of the constituent wave profiles). Thus, it would appear from this counterexample that the superposition principle is not generally valid for non-elliptic periodic waves. Moreover, an equation does not have to be 'integrable' (i.e. solvable by the inverse scattering transform) for the superposition principle to apply. For example, the so-called regularized long-wave (or Benjamin–Bona–Mahony) equation is not integrable [40], but its stationary periodic solutions are elliptic functions which can themselves be expressed as imbricate series [34].

The nonlinear superposition principle has been used to good effect by Zaitsev [25], Whitham [23], Miloh and Tulin [24] and Miloh [20] for finding periodic solutions of various nonlinear wave equations. These authors assume a stationary periodic solution in the form of an imbricate series of solitary-wave profiles and, by direct substitution, show that it solves the equation in question. It is evident that this direct approach has serious limitations. For instance, it is preferable that periodic stationary-wave solutions are known to exist *a priori* (though it is not strictly necessary). Also, the analytic form of the solitary wave must be known before the imbricate series can be constructed. Even so, after substituting the series into the equation, one may be left with an intractable identity to verify. To take but one example, Whitham [23], when applying the method to the KdV equation, is left to demonstrate that

$$\left[\sum_{n=-\infty}^{\infty} \operatorname{sech}^2(\xi-2n\sigma)\right]^2-\sum_{n=-\infty}^{\infty} \operatorname{sech}^4(\xi-2n\sigma)=B-A\sum_{n=-\infty}^{\infty} \operatorname{sech}^2(\xi-2n\sigma)$$

an astonishing result by any standard. That he was able to prove this identity (for suitable A and B) by elementary arguments is perhaps even more remarkable! A final disadvantage of this direct approach is that the imbricate series solution is usually taken to be real and may be a special case of a more general complex-valued periodic solution. The latter solution, as has been shown here for the ILW equation (section 11), may lead to other important (possibly periodic) solutions which would otherwise be 'lost'.

The advantage of the solution procedure adopted in the present article is that it yields complex-valued periodic solutions in terms of theta functions. It may then be possible to deduce the nonlinear superposition principle from the resulting solution using available theta identities. However, this approach suffers from the obvious disadvantage that it is restricted to those equations which can be solved via theta functions. To date, only those solutions which are expressible in terms of the 1D Riemann theta functions have been shown to accommodate the superposition principle. It is well known that the KdV, Boussinesq, Kadomtsev–Petviashvili, ILW and many other evolution equations admit solutions which can be expressed in terms of the multidimensional theta functions [see e.g. 26, 35, 41, 42]. These solutions have $N(>1)$ phase variables and can be thought of as the spatially periodic generalizations of the familiar N-soliton solutions (in the same way that a one-periodic wave ($N=1$) is a generalization of the solitary wave via the nonlinear superposition principle). They

† These are periodic solutions which can be expressed in terms of hyperelliptic functions, i.e. functions which are the inverse of hyperelliptic integrals [38].

have been variously referred to as 'finite band' or 'finite gap' solutions, 'N-periodic waves', 'N-cnoidal' or simply 'polycnoidal' waves (see the review article by Boyd [43]). *The problem of whether the nonlinear superposition principle can be extended* to polycnoidal waves remains an open one. The bicnoidal wave ($N=2$) of the KdV equation has been discussed at length by Boyd [44–46] who has shown that it can be approximated by the sum of two sech^2 solitary waves in the limit of large amplitudes (and when the wave crests are well separated compared with their widths). This, at least, suggests that the superposition principle may be valid for polycnoidal waves: however, balanced against this is the fact that these polycnoidal waves are hyperelliptic functions and, as noted above, the superposition principle fails for the hyperelliptic solutions of the quartic KdV.

This summary would not be complete without mention of a third procedure for obtaining imbricate series representations of periodic solutions. The method, first reported by Boyd [33] in connection with the KdV cnoidal wave, and subsequently developed in a series of articles by the same author [34, 44–46], uses the Poisson summation formula whereby a periodic function $u(x)$, with period $2L$, can be represented by an 'imbricate' series [32]

$$u(x)=u_0+\sum_{n=-\infty}^{\infty} S(x-2nL) \qquad u_0=\text{constant}. \tag{102}$$

The repeated 'pattern' function $S(x)$ must satisfy suitable decay conditions as $|x|\to\infty$ to ensure that the series converges. If $S(x)$ has the shape of a solitary wave, then we immediately recognize in (102) the nonlinear superposition principle. The method suffers from the disadvantage that the Fourier series of the periodic solution must be known before the series (102) can be determined. Furthermore, for those equations which are solvable via (1D) theta functions, there seems to be little advantage in using Boyd's technique. This is because the imbricate theta series, which are obtained by Poisson summation of the theta–Fourier series, can be deduced by simply applying Jacobi's modular transformation to the theta functions. Of course, the Poisson summation method may well be helpful for deriving imbricate series for those periodic stationary waves which cannot be expressed in terms of theta functions.

Finally, there is evidence to suggest that it may be possible to extend the superposition principle, albeit in a slightly modified form, to nonlinear wave equations which do not admit soliton-type solutions. For example, Parker [47] has shown that the periodic 'sawtooth' solution to the Burgers equation can be expressed as a superposition of identical, equally spaced Taylor [48] shock profiles. The Burgers equation is dissipative, and although the equation admits stationary waves with the Taylor shock profile these do not possess the 'clean interaction' property associated with soliton solutions. The nonlinear superposition principle as applied to the Burgers equation has been discussed at length in Parker [49]. It is interesting to speculate on which other nonlinear wave equations can be encompassed by the nonlinear superposition principle.

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